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LETTER TO THE EDITOR

More non-local symmetries of the KdV equation

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Abstract. The inverse of the usual recursion operator for the Korteweg–de Vries equation is applied to the Galilean symmetry. A new family of non-local symmetries results. A symmetry algebra isomorphic to the semidirect sum of the loop algebra over $\mathfrak{sl}(2,\mathbb{R})$ and a subalgebra of the Virasoro algebra is found.

The Korteweg–de Vries (KdV) equation $0 = u_t + u_{xxx} + 12uu_x$ has recursion operator [6]

$$\mathcal{R} = D_x^2 + 8u + 4u_x D_x^{-1}$$

and classical symmetry generators

$$\partial_x, \quad \partial_t, \quad -6t\partial_x - \tfrac{1}{2}\partial_u, \quad -\tfrac{3}{2}t\partial_t - \tfrac{1}{2}x\partial_x + u\partial_u$$

with respective characteristics [7]

$$-u_x, \quad -u_t, \quad 6tu_x - \tfrac{1}{2}, \quad u + \tfrac{1}{2}xu_x + \tfrac{3}{2}tu_t.$$

Repeatedly applying $\mathcal{R}$ to the zero characteristic yields $-u_x$, then u_t, followed by the famous infinite family of generalized symmetries [6]. In [3], $\mathcal{R}$ is applied repeatedly to the characteristic

$$\chi^{(1)} = 6tu_x - \tfrac{1}{2}$$

of the Galilean symmetry group, yielding $-4(u + \frac{1}{2}xu_x + \frac{3}{2}tu_t)$, the characteristic of the scaling symmetry group, and then an infinite family of (non-local) generalized symmetries. Recently, the fact that $\mathcal{R}^{-1}$ is also a recursion operator for the KdV equation has been used to generate three further infinite families of non-local symmetries [2]. These symmetries span a subalgebra, $\mathfrak{sl}(2,\mathbb{R}) \otimes \lambda \cdot \mathbb{R}[\lambda]$, of the loop algebra over $\mathfrak{sl}(2,\mathbb{R})$. One more infinite family of non-local symmetries of the KdV equation can be found by applying $\mathcal{R}^{-1}$ to the characteristic of the Galilean symmetry group.

Introduce the infinite-dimensional Wahlquist–Estabrook prolongation [8] of the KdV equation described by the system of first-order differential equations

$$(\Psi_n)_x = A_0\Psi_n + (1-\delta_n^0)A_1\Psi_{n-1} \tag{1}$$

$$(\Psi_n)_t = B_0\Psi_n + (1-\delta_n^0)B_1\Psi_{n-1} + (1-\delta_n^0)(1-\delta_n^1)B_2\Psi_{n-2} \tag{2}$$

together with the algebraic constraints

$$\sum_{i=0}^{n}(p_i s_{n-i} - q_i r_{n-i}) = \delta_0^n \tag{3}$$

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where δ_0^n is the Kronecker delta symbol and n ranges over the non-negative integers. Here

$$\Psi_n = \begin{pmatrix} p_n & q_n \\ r_n & s_n \end{pmatrix}$$

and

$$A_0 = \begin{pmatrix} 0 & 1 \\ -2u & 0 \end{pmatrix} \qquad A_1 = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}$$

$$B_0 = \begin{pmatrix} 2u_x & -4u \\ 2u_{xx} + 8u^2 & -2u_x \end{pmatrix} \quad B_1 = \begin{pmatrix} 0 & 4 \\ -4u & 0 \end{pmatrix} \quad B_2 = \begin{pmatrix} 0 & 0 \\ -4 & 0 \end{pmatrix}.$$

The symmetry algebra of this system contains a subalgebra with basis $\{\boldsymbol{u}_m, \boldsymbol{v}_m, \boldsymbol{w}_m : m \in \mathbb{Z}\}$ and commutators

$$[\boldsymbol{u}_{m_1}, \boldsymbol{u}_{m_2}] = [\boldsymbol{v}_{m_1}, \boldsymbol{v}_{m_2}] = [\boldsymbol{w}_{m_1}, \boldsymbol{w}_{m_2}] = 0$$
$$[\boldsymbol{u}_{m_1}, \boldsymbol{v}_{m_2}] = 2\boldsymbol{v}_{m_1+m_2} \quad [\boldsymbol{u}_{m_1}, \boldsymbol{w}_{m_2}] = -2\boldsymbol{w}_{m_1+m_2} \quad [\boldsymbol{v}_{m_1}, \boldsymbol{w}_{m_2}] = -\boldsymbol{u}_{m_1+m_2}.$$

Details can be found in [2].

The description of $\mathcal{R}^{-1}$ follows the approach of [1]. If Q is the characteristic of a (non-local) generalized symmetry of the KdV equation and $\{F, G, H\}$ satisfy

$$D_x(F) = p_0^2 Q \qquad D_t(F) = (-p_0^2 D_x^2 + 2p_0 r_0 D_x - 2(4up_0^2 + r_0^2))(Q)$$
$$D_x(G) = p_0^{-2} F \quad D_t(G) = (-D_x + 2p_0^{-1} r_0)(Q) - 4up_0^{-2} F$$
$$D_x(H) = p_0^{-2} G \quad D_t(H) = -p_0^{-2} Q - 4up_0^{-2} G$$

then $\mathcal{R}^{-1}(Q) = G + 2p_0 r_0 H$ is also the characteristic of a (non-local) generalized symmetry of the KdV equation. The notation $\mathcal{R}^{-1}(Q)$ is motivated by the observation that $\mathcal{R}(G + 2p_0 r_0 H) = Q + au_x$, with a an arbitrary constant. For example,

$$\chi^{(2)} = -4\mathcal{R}^{-1}(\chi^{(1)}) = 2(q_1 r_0 - p_1 s_0)$$

is the characteristic of the non-local symmetry generator

$$\boldsymbol{x}_2 = \chi^{(2)} \partial_u + \sum_{i=0}^{\infty} (M_i^{(2)} \partial_{p_i} + N_i^{(2)} \partial_{q_i} + D_x(M_i^{(2)}) \partial_{r_i} + D_x(N_i^{(2)}) \partial_{s_i}).$$

The functions

$$M_i^{(2)} = -(i+2) p_{i+2} + p_0(q_1 r_{i+1} - p_{i+1} s_1) + q_0(p_{i+1} r_1 - p_1 r_{i+1})$$

and

$$N_i^{(2)} = -(i+2) q_{i+2} - p_0(q_{i+1} s_1 - q_1 s_{i+1}) - q_0(p_1 s_{i+1} - q_{i+1} r_1)$$

are found by requiring invariance of equations (1) to (3) under the action of pr $\boldsymbol{x}_2$.

The family of symmetries $\{\mathcal{R}^{-n}(\chi^{(1)}) : n = 1, 2, \ldots\}$ could be generated using repeated applications of $\mathcal{R}^{-1}$, following the method of [2]. Here, however, the underlying Lie algebra is such that this process can be avoided. Notice that the Galilean symmetry generator prolongs to

$$\boldsymbol{x}_1 = -6t\partial_x - \tfrac{1}{2}\partial_u - \sum_{i=0}^{\infty} (i+1)(p_{i+1}\partial_{p_i} + q_{i+1}\partial_{q_i} + r_{i+1}\partial_{r_i} + s_{i+1}\partial_{s_i}).$$

For each integer $n \geq 3$,

$$\begin{aligned}\boldsymbol{x}_n &= [\boldsymbol{x}_1, \boldsymbol{x}_{n-1}]/(2-n) \\ &= \chi^{(n)}\partial_u + \sum_{i=0}^{\infty}(M_i^{(n)}\partial_{p_i} + N_i^{(n)}\partial_{q_i} + D_x(M_i^{(n)})\partial_{r_i} + D_x(N_i^{(n)})\partial_{s_i})\end{aligned}$$

must be a non-local symmetry generator of the KdV equation. One finds that

$$\begin{aligned}\chi^{(n)} &= 2\sum_{i=1}^{n-1} i(q_i r_{n-1-i} - p_i s_{n-1-i}) \\ M_i^{(n)} &= -(i+n)p_{i+n} \\ &\quad + \sum_{j=0}^{n-2} p_j \sum_{k=1}^{n-1-j} k(q_k r_{i+n-j-k} - p_{i+n-j-k}s_k) \\ &\quad + \sum_{j=0}^{n-2} q_j \sum_{k=1}^{n-1-j} k(p_{i+n-j-k}r_k - p_k r_{i+n-j-k}) \\ N_i^{(n)} &= -(i+n)q_{i+n} \\ &\quad - \sum_{j=0}^{n-2} p_j \sum_{k=1}^{n-1-j} k(q_{i+n-j-k}s_k - q_k s_{i+n-j-k}) \\ &\quad + \sum_{j=0}^{n-2} q_j \sum_{k=1}^{n-1-j} k(q_{i+n-j-k}r_k - p_k s_{i+n-j-k})\end{aligned}$$

for all integers $i \geq 0$ and $n \geq 2$. A further symmetry of the prolonged system is

$$\boldsymbol{x}_0 = -\tfrac{3}{2}t\partial_t - \tfrac{1}{2}\boldsymbol{x}\partial_x + \boldsymbol{u}\partial_u - \sum_{i=0}^{\infty}(\mathrm{i}p_i\partial_{p_i} + iq_i\partial_{q_i} + (\mathrm{i}-\tfrac{1}{2})r_i\partial_{r_i} + (\mathrm{i}-\tfrac{1}{2})s_i\partial_{s_i}).$$

The Wahlquist–Estabrook prolongation of the KdV equation described by equations (1) to (3) thus admits four distinct infinite families of symmetry generators. These symmetries generate a remarkable Lie algebra. It has basis $\{\boldsymbol{u}_m, \boldsymbol{v}_m, \boldsymbol{w}_m, \boldsymbol{x}_n : m, n \in \mathbb{Z},\ n \geq 0\}$ and commutator table

	$\boldsymbol{u}_{m_2}$	$\boldsymbol{v}_{m_2}$	$\boldsymbol{w}_{m_2}$	$\boldsymbol{x}_{n_2}$
$\boldsymbol{u}_{m_1}$	0	$2\boldsymbol{v}_{m_1+m_2}$	$-2\boldsymbol{w}_{m_1+m_2}$	$-m_1\boldsymbol{u}_{m_1-n_2}$
$\boldsymbol{v}_{m_1}$		0	$-\boldsymbol{u}_{m_1+m_2}$	$-m_1\boldsymbol{v}_{m_1-n_2}$
$\boldsymbol{w}_{m_1}$			0	$-m_1\boldsymbol{w}_{m_1-n_2}$
$\boldsymbol{x}_{n_1}$				$(n_1-n_2)\boldsymbol{x}_{n_1+n_2}$

That is, the KdV equation admits a symmetry algebra isomorphic to the semidirect sum of the loop algebra over $\mathfrak{sl}(2,\mathbb{R})$ and a subalgebra of the Virasoro algebra.

Notice that $\mathcal{R} : \chi^{(n)} \mapsto -4\chi^{(n-1)}$ for all positive integers n so that, as claimed above, the Virasoro algebra is generated by the action of $\mathcal{R}^{-1}$ on the Galilean symmetry. The results of this letter, when combined with those of [2], [3] and [6], yield all the symmetries found by applying the recursion operator (and its inverse) to classical symmetries of the KdV equation.

It is conjectured that similarly rich non-local symmetry structures can be associated with other integrable differential equations. Lou [5] has begun analysing the non-local symmetry structure of the Sawada–Kotera equation by inverting that equation's recursion operator. Kiso [4] has constructed an algebra $\mathfrak{sl}(2, \mathbb{C}) \otimes \mathbb{C}[\lambda, \lambda^{-1}]$ of symmetries of the AKNS hierarchy using an elegant Lie algebraic construction which avoids the use of recursion operators altogether.

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